Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis I

Mid-term Examination | Date : Sept. 7, 2016
Maximum marks: 100 | Time: 3 hours

Here the set of natural numbers $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}$ and the set of real numbers is denoted by $\mathbb{R}$.

1. Let S be a non-empty finite set and let $g : S \to S$ be a function. Show that there exists $x \in S$ and $n \in \mathbb{N}$ such that
$$g^n(x) = x,$$
where $g^n = g \circ g \circ \cdots \circ g$ (n times). [15]

2. Prove or disprove the claim that $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined by
$$f(m, n) = 2^{m-1}(2n - 1),$$
is a bijection. [15]

3. Show that given any two numbers $x, y \in \mathbb{R}$ with $x < y$ there exists a rational number r such that $x < r < y$. [15]

4. Let $\{b_n\}_{n\geq 1}$ be a sequence of non-zero real numbers converging to a non-zero real number c. Show that the sequence $\{\frac{1}{b_n}\}_{n\geq 1}$ converges to $\frac{1}{c}$. [15]

5. Define lim inf and lim sup of bounded sequences. Show that a bounded sequence $\{x_n\}_{n\geq 1}$ is convergent if and only if
$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$
[15]

6. Show that a series of real numbers $\sum_{n=1}^{\infty} a_n$ is convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent and the converse is not true. [15]

7. Let z be the real number with binary expansion:
$$z = 0.b_1 b_2 b_3 \ldots,$$
where $b_k = 1$ if $k = 3n + 1$ for some natural number n, and $b_k = 0$ otherwise. Compute the decimal expansion of z. [15]